Abstract
An approximate scattering matrix for an interface separating two arbitrary anisotropic half-spaces is obtained. The matrix depends on perturbation of elastic coefficients, densities, and directions of wave propagation. Compact linearized formulas for the reflection/transmission coefficients are derived for a weak-contrast interface and an anisotropic background. In case of an isotropic background the PP reflection/transmission coefficients are reduced to simple analytic equations expressed in terms of independent linear combinations of perturbations of normalized elastic coefficients and densities, which can be determined from observed data.

Introduction
Approximate formulas for the reflection/transmission (R/T) coefficients are widely used to invert seismic data into the physical properties of a media, especially in amplitude-versus-offset (AVO) analysis and amplitude-versus-azimuth (AVA) analysis, and to study the behavior of these coefficients in the neighborhood of a given direction.


Theory
Following these studies, let us consider a medium with two anisotropic half-spaces separated by plane \( x_3 = 0 \) and denote the densities and the elastic parameters at the both half-spaces as \( \rho^{(i)} \) and \( \lambda^{(i)}_{pq;rs} \), \( i=1,2 \), respectively.

Plane elastic waves in the half-spaces \( i=1,2 \) with a common frequency \( \omega \) and an identical projection of a slowness vectors \( s^{(i)}_{\alpha} = (s_1^{(i)}, s_2^{(i)}, s_3^{(i)}) \) onto the plane \( x_3 = 0 \) are the sum of the waves with different modes \( \alpha \) and can be expressed in the following way:

\[
u^{(i)}(r,t) = \sum_{\alpha} b^{(i)}_{\alpha}(r,t) \exp\left(j \omega (s \cdot r - t)\right),
\]

where \( a^{(i)}_{\alpha} \) is a unit polarization vector, \( b^{(i)}_{\alpha} \) is a scalar amplitude. The subscript \( \alpha \) denotes a type of waves: \( \alpha=1,2,3 \) correspond to the down-going \( qP \), \( S_1 \), \( S_2 \) waves, while \( \alpha=4,5,6 \) correspond to up-going \( qP \), \( S_1 \), \( S_2 \) waves, respectively.

Let \( C^{(i)mn}_{pq} \) denote a 3x3 matrix whose elements are the elastic parameters:

\[
C^{(i)mn}_{pq} = \lambda^{(i)}_{np;mq}.
\]

An \( \alpha \)-type wave presses per unit area of the interface \( x_3 = 0 \) with the force

\[
f^{(i)}_a = -j \omega (s \cdot r - t),
\]

where

\[
I_{\alpha}^{(i)} = -s_1^{(i)} C^{(i)31}_{31} + s_2^{(i)} C^{(i)32}_{32} + s_3^{(i)} C^{(i)33}_{33} a^{(i)}_{\alpha},
\]

(2)

Let \( v^{(i)}_a \) and \( n^{(i)}_a \) denote 6-dimensional column and row vectors respectively, determined in the following way:

\[
v^{(i)}_a = \begin{bmatrix} a^{(i)}_1 \\ a^{(i)}_2 \\ v^{(i)}_1 \\ v^{(i)}_2 \end{bmatrix}, \quad n^{(i)}_a = \begin{bmatrix} a^{(i)T}_1 \\ a^{(i)T}_2 \end{bmatrix},
\]

where \( T \) stands for transposition.

In each of the half-spaces \( i=1,2 \) let \( \{ a^{(i)}_a, \rho^{(i)} \} \) are the eigenpairs of symmetrical Christoffel matrices \( G^{(i)}_a = \sum_{p,q=1,2} s_p s_q C^{(i)p}_{pq} + \sum_{p=1,2} s_3 s_p C^{(i)p}_{3p} + s_3 s_3 C^{(i)33}_3, \) which depend on the wave type, \( \alpha : \)

\[
G^{(i)}_a a^{(i)}_a = \rho^{(i)} a^{(i)}_a.
\]

(3)
Let \( C^{(i)}_{33}, \quad i=1,2 \) be nonsingular. This is usually the case since these matrixes possess relatively great elements \( \lambda^{(i)}_{13,13}, \lambda^{(i)}_{23,23}, \lambda^{(i)}_{33,33} \) on their principal diagonals and relatively small ones elsewhere.

Equation (2) can be expressed as
\[
C^{(i)-1}_{33} \left( s_3 C^{(i)}_{31} + s_2 C^{(i)}_{32} \right) a^{(i)}_a + C^{(i)-1}_{33} \lambda^{(i)}_a = -s_3^{(i)} a^{(i)}_a. \tag{4}
\]

Substituting (2) and (4) in (3) yields:
\[
\sum_{p,q=1,2} s_p s_q C^{(i)}_{33} C^{(i)}_{3q} - C^{(i)}_{33} C^{(i)}_{3q} = C^{(i)-1}_{33} \lambda^{(i)}_a + C^{(i)}_{33} C^{(i)-1}_{33} \lambda^{(i)}_a = -s_3^{(i)} \lambda^{(i)}_a. \tag{5}
\]

Jointly, equations (4) and (5) demonstrate that the vectors \( v^{(i)}_a \) and \( n^{(i)}_a \) are the right and left eigenvectors of 6x6 matrix
\[
M^{(i)} = \begin{pmatrix} A^{(i)} & C^{(i)-1}_{33} \\ B^{(i)} & A^{(i)} \end{pmatrix}
\]
associated with their eigenvalues, \( s^{(i)}_a \), i.e.
\[
M^{(i)} v^{(i)}_a = s^{(i)}_a v^{(i)}_a \quad \text{and} \quad n^{(i)}_a M^{(i)} = s^{(i)}_a n^{(i)}_a,
\]
where
\[
A^{(i)} = C^{(i)}_{33} \left( s_3 C^{(i)}_{31} + s_2 C^{(i)}_{32} \right), \quad B^{(i)} = \sum_{p,q=1,2} s_p s_q C^{(i)}_{33} C^{(i)}_{3q} - C^{(i)}_{33} C^{(i)}_{3q} + \rho^{(i)} I.
\]

Unlike the Christoffel matrixes \( C^{(i)}_a \), the matrixes \( M^{(i)} \) do not contain the component \( s^{(i)}_a \) of the slowness vector \( s^{(i)}_a \). For this reason, they are referred not to a concrete wave mode, but to wave packets as a whole.

Let \( E^{(i)} \) be a matrix consisting of the column eigenvectors \( v^{(i)}_a \) of the matrix \( M^{(i)} \). From (7) it follows that the inverse matrix \( \left( E^{(i)} \right)^{-1} \) consists of the following row eigenvectors:
\[
w^{(i)}_a = \frac{n^{(i)}_a}{n^{(i)}_a \cdot v^{(i)}_a} = \frac{n^{(i)}_a}{21^{(i)} a^T a^{(i)}}.
\]

Let \( X = (v^{(i)}_1, v^{(i)}_2, v^{(i)}_3, v^{(i)}_4, v^{(i)}_5, v^{(i)}_6) \) and \( Y = (v^{(i)}_1, v^{(i)}_2, v^{(i)}_3, v^{(i)}_4, v^{(i)}_5, v^{(i)}_6) \) be the matrixes consisting of the column eigenvectors of matrices \( M^{(i)}, i=1,2 \), for all the types of plane waves coming to and leaving the interface, respectively, and \( S \) be the elastic scattering matrix containing all the R/T coefficients.

If \( Y \) is nonsingular, its columns form a basis in a 6-dimensional vector space. In this case the scattering matrix \( S \) can be found as \( S = Y^{-1} X \) (Aki and Richards, 1980). The calculation of \( S \) is thus reduced to calculation of the eigenvectors of the matrixes \( M^{(i)} \).

If the matrix \( M^{(i)} \) undergoes a perturbation, \( \Delta M^{(i)} \), caused by deviations of the elastic coefficients \( \lambda^{(i)}_{np,q} \in C^{(i)}_{pq} \), the components \( s_1 \) and \( s_2 \) of slowness vectors \( s^{(i)}_a \), and densities \( \rho^{(i)} \), the scattering matrix \( S \) obtains the following perturbation
\[
\Delta S = Y^{-1} \cdot \Delta X - Y^{-1} \cdot \Delta Y \cdot S. \tag{8}
\]

The matrices \( X \) and \( Y \) can be expressed as follows:
\[
X = \begin{pmatrix} E^{(1)}_{11} & E^{(1)}_{12} \\ E^{(2)}_{11} & E^{(2)}_{12} \end{pmatrix}, \quad Y = \begin{pmatrix} E^{(2)}_{11} & E^{(2)}_{12} \\ E^{(1)}_{21} & E^{(1)}_{22} \end{pmatrix}, \quad \text{where} \quad E^{(i)} = \begin{pmatrix} E^{(i)}_{11} & E^{(i)}_{12} \\ E^{(i)}_{21} & E^{(i)}_{22} \end{pmatrix}
\]
is splitting the matrix \( E^{(i)} \) into the four 3x3-submatrixes \( E^{(i)}_{pq} \) (i, p, q=1,2).

According to the perturbation theory, if the matrix \( M^{(i)} \) has eigenvalues \( s^{(i)}_p \) of index 1, the first-order perturbation of the eigenvalues \( \delta s^{(i)}_p \) and eigenvectors \( \delta v^{(i)}_p \) of this matrix can be calculated as:
\[
\delta s^{(i)}_p = w^{(i)}_p \Delta M^{(i)} v^{(i)}_p, \tag{10}
\]
\[
\delta v^{(i)}_p = d^{(i)}_p v^{(i)}_p + \sum_{q \neq p} d^{(i)}_{pq} v^{(i)}_q, \tag{11}
\]
where
\[
d^{(i)}_p = \frac{w^{(i)}_p \Delta M^{(i)} v^{(i)}_p}{s^{(i)}_p - s^{(i)}_q}, \quad \text{when} \quad p \neq q \quad \text{and} \quad s^{(i)}_p \neq s^{(i)}_q, \tag{12}
\]
\[
d^{(i)}_p = 0, \quad \text{when} \quad p \neq q \quad \text{and} \quad s^{(i)}_p = s^{(i)}_q, \tag{13}
\]
\[
d^{(i)}_{pq} = \sum_{p \neq q} d^{(i)}_{pq} a^{(i)}_q \cdot a^{(i)}_p.
\]

Note, equation (13) follows from the condition of normalization: \( |a^{(i)}_p + \delta a^{(i)}_p| = 1 \).
If the matrix $M^{(i)}$ has a multiple eigenvalue, $s_{iP}$, the associated eigenvectors, $v_{p}^{(i)}, \ldots v_{n}^{(i)}$, should be selected to satisfy the equations $w_{p}^{(i)} \Delta M v_{p}^{(i)} = 0$, with $m \neq n$.

Let $\Delta G_{qp}^{(i)}$ denote a matrix of the expression

$$
G_{qp}^{(i)} = \sum_{n=1,2} s_{n}^{(i)} s_{n} C_{mn}^{(i)} + \sum_{n=1,2} C_{mn}^{(i)} s_{n} + C_{33}^{(i)} s_{3}^{(i)} - \rho I ,
$$

that emerges due to some changes in the elastic coefficients, the components $s_{i}$ and $s_{2}$ of the slowness vector, and the densities $\rho^{(i)}$, with the values $s_{3P}$ and $s_{3q}$ being invariable. Substituting $\rho^{(i)} = -C_{33}^{(i)} (A^{(i)} + s_{2}^{(i)}) a_{q}^{(i)}$ and $B^{(i)} = A^{(i)} C_{33}^{(i)} A^{(i)} - \sum_{n=1,2} C_{mn}^{(i)} s_{n} + \rho I$ in the expression $n_{q}^{(i)} \Delta M v_{p}^{(i)}$ yields

$$
n_{q}^{(i)} \Delta M v_{p}^{(i)} = a_{q}^{(i)} \Delta G_{qp}^{(i)} a_{q}^{(i)} .
$$

The similar equation, in the tensor notation, was recently obtained by Klimes (2003).

Let us introduce a matrix $D^{(i)} = \rho^{(i)} \Delta G_{qp}^{(i)} a_{q}^{(i)}$. Then equation (11) can be rewritten in a compact form:

$$
\Delta E^{(i)} = E^{(i)} \cdot D^{(i)} .
$$

Using (8), (9), and (16), the following equations for the perturbations of the matrices $X$, $Y$ and $S$ can be obtained:

$$
\Delta X = X \begin{pmatrix} D_{11}^{(i)} & 0 \\ 0 & D_{22}^{(i)} \end{pmatrix} + Y \begin{pmatrix} 0 & D_{12}^{(i)} \\ D_{21}^{(i)} & 0 \end{pmatrix} ;
\Delta Y = X \begin{pmatrix} 0 & D_{12}^{(i)} \\ D_{21}^{(i)} & 0 \end{pmatrix} + Y \begin{pmatrix} D_{11}^{(i)} & 0 \\ 0 & D_{22}^{(i)} \end{pmatrix} ;
\Delta S = \begin{pmatrix} 0 & D_{12}^{(i)} \\ D_{21}^{(i)} & 0 \end{pmatrix} + S \begin{pmatrix} D_{11}^{(i)} & 0 \\ 0 & D_{22}^{(i)} \end{pmatrix} - \begin{pmatrix} D_{11}^{(i)} & 0 \\ 0 & D_{22}^{(i)} \end{pmatrix} S .
$$

In a particular case of half-planes with similar physical properties, the medium can be considered as a perturbation of the homogeneous background. Since $S$ for homogeneous media is an identity matrix, expression (17) for a weak-contrast interface becomes much simpler:

$$
\Delta S = \begin{pmatrix} D_{11}^{(i)} - D_{12}^{(i)} & -D_{12}^{(i)} + D_{11}^{(i)} \\ -D_{21}^{(i)} + D_{22}^{(i)} & D_{22}^{(i)} - D_{21}^{(i)} \end{pmatrix} .
$$

Equation (18) demonstrates how a perturbation of the scattering matrix for a weak-contrast interface depends on differences in the elasticity and density of the half-spaces and a perturbation of the slowness.

The dependence of the PP reflection/transmission coefficients and qP-wave phase velocities squared on a perturbation of a homogeneous isotropic background can be written as:

$$
k_{p}(n_{1}, n_{2}) = \frac{f_{1}}{2} - \frac{f_{2}}{2} + \frac{n_{2}^2}{4} z_{33} ;
$$

$$
k_{p}(n_{1}, n_{2}) = 1 + \frac{1}{4} \left( 2 + \frac{1}{n_{4}} \right) f_{4} + \frac{2n_{2}^2 - 1}{2} f_{2} + \frac{n_{3}^2(2n_{2}^2 - 3)}{4} z_{33} + 2n_{3}f_{3} ;
$$

$$
v_{p}^{2}(n_{1}, n_{2}) = \alpha^{2} \left( 1 + f_{4} + 2n_{2}^2 f_{2} + n_{3}^2 n_{33} + 4n_{1}f_{1} + f_{3} \right) ,
$$

where $a_{q} = \Delta c_{q} / (\rho c_{z})$, $n_{3}(n_{1}, n_{2}) = \sqrt{1 - n_{1}^2 - n_{2}^2}$, $f_{1} = a_{23}n_{1} + a_{34}n_{2}$, $f_{2} = z_{34}n_{3}^2 + 2z_{45}n_{1}n_{2} + z_{46}n_{2}^2$, $f_{3} = z_{13}n_{1}^2 + z_{56}n_{1}^2 + z_{64}n_{1}^2 + z_{23}n_{2}^2$, $f_{4} = a_{21}n_{1}^3 + 4a_{41}n_{1}n_{2}^2 + 2z_{45n_{1}}n_{2}^2 + 4a_{36}n_{3}^2 + z_{23}n_{3}^2$, $z_{33} = a_{33} + r_{6}$, $z_{44} = 2a_{44} - e a_{23} + r_{6}$, $z_{55} = 2a_{55} - e a_{13} + r_{6}$, $z_{66} = 2a_{66} + a_{12}$, $z_{45} = 2a_{45} - e a_{36}$, $z_{23} = a_{23} - a_{34}$, $z_{14} = a_{14} - a_{34}$, $z_{24} = 2a_{44} - e a_{23} + r_{6}$, $z_{46} = 2a_{66} - a_{14} + a_{44}$, $z_{64} = 2a_{66} - a_{14} + a_{44}$; $r = \frac{4f_{1}}{\alpha^{2} - 1} \frac{\Delta \rho}{\rho}$, $r_{6} = \frac{2}\Delta \rho \rho$ and $e = 1$ for reflected waves, $r = \frac{\Delta \rho}{\rho}$, $r_{6} = \frac{2\Delta \rho}{\rho}$ and $e = -1$ for transmitted waves, $r = 0$, $r_{6} = 0$ and $e = -1$ for phase velocities;
α and β are P- and S-wave velocities, ρ - is the density of a referent isotropic medium, \(\Delta c_{ij}\) are deviations of the elastic coefficients, \(n = (n_1, n_2, n_3)\) is a unit direction vector.

In expressions (19) - (21), the factors at \(a_{ij}\) and \(z_{ij}\) are linearly independent functions. For this reason, the parameters \(a_{ij}\) and \(z_{ij}\) from the above equations can be determined uniquely using a sufficiently large number of reflection/transmission coefficients and phase velocities. Analysis this equations allows us to draw a conclusion that only the elastic parameters and their linear combinations from Table 1 can be determined independently and uniquely.

<table>
<thead>
<tr>
<th>Table 1: Elastic parameters recoverable from observed variables (k_a, k_T, v_{qp}^2)</th>
<th>(a_{11})</th>
<th>(a_{22})</th>
<th>(a_{11} + r_o)</th>
<th>(a_{16})</th>
<th>(a_{36})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k_a)</td>
<td>(2a_{44} - a_{23} + r)</td>
<td>(2a_{55} - a_{13} + r)</td>
<td>(2a_{45} - a_{36})</td>
<td>(2a_{46} + a_{12})</td>
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<tr>
<td>(k_T)</td>
<td>(2a_{44} + a_{23} + r)</td>
<td>(2a_{55} + a_{13} + r)</td>
<td>(2a_{45} + a_{36})</td>
<td>(a_{24} - a_{34})</td>
<td>(a_{15} - a_{35})</td>
</tr>
<tr>
<td>(v_{qp}^2)</td>
<td>(a_{34})</td>
<td>(a_{35})</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Equations (19)-(21) for various directions \(n_i\) represent an overdetermined system of linear equations in the unknowns from Table 1. Let \(Fx = b\) be a matrix representation of the above system of equations.

In Figure 1, the eigenvalues of the matrix \((F^T)\) are displayed for various combinations of equations (19)-(21). The eigenvalues were normalized and arranged in a non-ascending order. The directions \(n_i\) uniformly covered an incidence interval \(0 < \theta < \pi / 4\) and an azimuth interval \(0 < \phi < 2\pi\). For simplicity, it was supposed that \(\Delta \rho = 0\).

![Figure 1: Normalized eigenvalues of the matrix \((F^T)\).](image)

Conclusions

Analysis of Table 1 and Figure 1 allows us to draw a conclusion that 9, 13 and 15 elastic parameters or their linear combinations can be determined with the use of PP reflection coefficients (R), PP transmission coefficients (T) and qP-wave velocities (V), respectively. Joint utilization of the observed data increases the number of recoverable parameters: (R+T) – 16; (R+V) – 18; (T+V) – 15 (the same parameters, that are determined using qP-wave velocities). In case of R+V it is impossible to restore deviations of the elastic coefficients \(\Delta c_{41}\) and \(\Delta c_{16}\) and to separate the linear combination \(6c_{21} + 2\Delta c_{66}\).

References